Generalization of First Conway-Gordon Theorem

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Outline

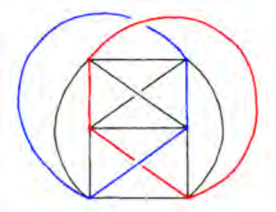
The Theorem

Theorem (The First Conway-Gordon Theorem)

Every Spatial Embedding of K_6 contains a non-trivial link

Definition

A **Spatial Embedding** of a graph G is the image of a injective continuous map $f : G \to \mathbb{R}^3$



Outline of Conway-Gordon Proof

- Define the invariant $\Omega(G) = \sum \text{lk}(\alpha, \beta) \pmod 2$, where the sum is over all unordered pairs of Hamiltonian pairs of cycles in G.
- Show that Ω is invariant under isotopy and crossing changes.
- Apply lemma that states that any spatial embedding can be changed to any other using only isotopy and crossing changes.
- Show that $\Omega = 1$ for a specific embedding of K_6.
- Conclude that $\Omega = 1$ for all spatial embeddings of K_6, and thus all embeddings of K_6 contain a nontrivial Hamiltonian link.

Generalizing to K_n

Most of the proof does not rely on the graph being K_6:

- Define the invariant $\Omega(G) = \sum \text{lk}(\alpha, \beta) \pmod 2$, where the sum is over all unordered pairs of Hamiltonian pairs of cycles in G.
- **Show that Ω is invariant under** isotopy and **crossing changes**.
- Apply lemma that states that any spatial embedding can be changed to any other using only isotopy and crossing changes.
- **Show that $\Omega = 1$ for a specific embedding of K_n.**

The Results of the Generalization

The following theorems related to the generalization of the first Conway-Gordon theorem were given by Kazakov and Korablev:

Theorem

For any two spatial embeddings G_n', G_n'' of K_n, $n \geq 6$, $\Omega(G_n') = \Omega(G_n'')$.

Theorem

Let G_n be a spatial embedding of K_n. Then $\Omega(G_n) = 1$ if $n = 6$, and $\Omega(G_n) = 0$ if $n > 6$.

The First Theorem

Theorem

For any two spatial embeddings G_n', G_n'' of K_n, $n \geq 6$, $\Omega(G_n') = \Omega(G_n'')$.

Where

$$\Omega(G) = \sum \text{lk}(\alpha, \beta) \pmod 2$$

is the sum is over all unordered pairs of Hamiltonian pairs of cycles in G.

Idea of Proof

This proof relies on the same lemma used in the Conway-Gordon proof:

Lemma

Let G' and G'' be spatial embeddings of the same graph. Then, G' can be transformed to G'' by a series of crossing changes and isotopies

All that remains is to show that Ω is invariant under isotopies and crossing changes.
Since Ω is the sum of linking numbers (which are invariant under isotopy) clearly Ω is invariant under isotopy.

The Proof

Once again, to show that Ω is unaffected over crossing changes. There are three cases, but not really:

1. The crossing is with and edge and itself
2. The crossing is between two adjacent edges
3. The crossing is between two-nonadjacent edges

However, the first two cases can be discarded, as any such crossing change has no effect on any linking number, and thus has no effect on Ω

The Proof, Cont.

Thus, we need only consider when an edge crosses with another, non-adjacent edge, call them A and B. As before, for a given link $L = (L_1, L_2)$, $A \subset L_1$ and $B \subset L_2$, it's linking number changes by 1 after the crossing change. So,

$$\Omega = \Omega' + \sum_{(A,B)\subset(L_1,L_2)} 1 \pmod 2$$

And we need noly show that the number of such crossings is even. To this end, for a given link (L_1, L_2), break it down as follows, where a_1, a_2 are the vertices of A, and b_1, b_2 are the vertices of B

$$L_1 = a_1 \mapsto a_2 \mapsto v_1 \mapsto \cdots \mapsto v_n \mapsto a_1$$
$$L_2 = b_1 \mapsto b_2 \mapsto w_1 \mapsto \cdots \mapsto w_m \mapsto b_1$$

The Great Finale

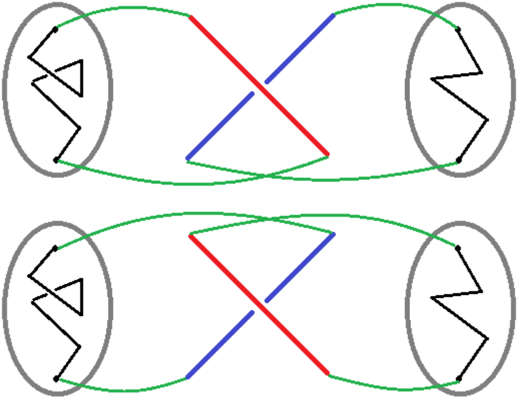
$$L_1 = a_1 \mapsto a_2 \mapsto v_1 \mapsto \cdots \mapsto v_n \mapsto a_1$$
$$L_2 = b_1 \mapsto b_2 \mapsto w_1 \mapsto \cdots \mapsto w_m \mapsto b_1$$

From this description of L_1 and L_2, define a link

$$N_1 = a_1 \mapsto a_2 \mapsto w_1 \mapsto \cdots \mapsto w_n \mapsto a_1$$
$$N_2 = b_1 \mapsto b_2 \mapsto v_1 \mapsto \cdots \mapsto v_m \mapsto b_1$$

This defines a map of order two between links, and because $(N_1, N_2) \neq (L_1, L_2)$, it is a pairing between the set of links so that $A \subset L_1, B \subset L_2$. Thus, Ω is invariant over any spatial embedding of K_n

An Illustrated Diagram



Idea of the Proof

Theorem

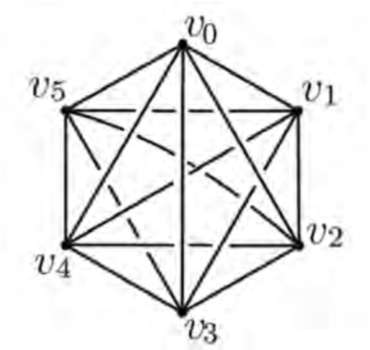
Let G_n be a spatial embedding of K_n. Then $\Omega(G_n) = 1$ if $n = 6$, and $\Omega(G_n) = 0$ if $n > 6$.

Because of theorem one, we now know that Ω is independent of spatial embedding, so we need only calculate it for a specific embedding given n. To this end, we chose the following embedding:

1. Label the vertices of K_n $v_0, v_1, v_2, \ldots v_{n-1}$
2. Place them in $\mathbb{R}^3$ so that they project down to a regular n-gon
3. Fix v_0, then place the vertex v_i i units lower than v_0 in the z-direction

Chosen Embedding example

Projected down to $\mathbb{R}^2$, the embedding looks something like this:



In general, two edges, call them $e = (v_i, v_j)$ and $f = (v_k, v_l)$, $i < j$, $k < l$ intersect iff either $l < j < k < i$ or $j < l < i < k$.

Introducing Destabilization

For ease of reference, call the set of (Unordered) disjoint pairs of Hamiltonian Cycles in K_n Γ_n. We're trying to compute a huge sum, being Ω, so the general strategy will be to partition Γ in to easy to compute chunks.

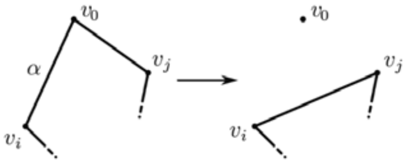
Definition

$\Delta(v_0)$ is the set of all disjoint Hamiltonian cycles (α, β) so that one of the cycles is a "triangle" containing v_0.

Destabilization is going to be defined of $\Gamma_n \setminus \Delta(v_0)$.

Destabilization

Let $(\alpha, \beta) \in \Gamma_n \setminus \Delta(v_0)$, and WLOG, assume $v_0 \in \alpha$. Then, α has at least four vertices, two of which are neighboring v_0, call them v_i and v_j. The destabilization of this pair results in a new pair (α', β), where the edges (v_i, v_0) and (v_0, v_j) are removed and replaced with (v_i, v_j)



Notice we are **not** left with a something in Γ_n. It is however, within Γ_{n-1} when considering $K_n \setminus \{v_0\}$.

Dealing with $\Gamma_n \setminus \Delta(v_0)$

Lemma

Let $G_n \subset \mathbb{R}^3$ be a spatial complete graph with $n \geq 6$ vertices. Then

$$\sum_{(\alpha,\beta)\in\Gamma_n\setminus\Delta(v_0)} \mathrm{lk}_2(\alpha,\beta) \equiv (n+1) \sum_{[(\alpha,\beta)]} \mathrm{lk}_2(\gamma,\delta) \pmod 2,$$

Where the last sum is taken over all equivalence classes of Hamiltonian pairs of cycles in $\Gamma_n \setminus \Delta(v_0)$ and the pair (γ,δ) is obtained by destabilization of (α,β) along the vertex v_0.

Proof of Lemma 1

Let (δ, γ) be the image of some destabilization along the vertex v_0. Any element that reduces to it (i.e. elements of the set $[(\alpha, \beta)]$), can be obtained as follows:

There are $n-1$ places to "add back" the vertex v_0. For any edge $e \in (\delta, \gamma)$, with end points v_i and v_j, we can replace it with the two edges (v_i, v_0) and (v_0, v_j).

The question is now how $\text{lk}_2(\alpha', \beta')$ and $\text{lk}_2(\delta, \gamma)$ relate, where $(\alpha', \beta') \in [(\alpha, \beta)]$

Proof of Lemma 1, Cont.

With out loss of generality, assume that $v_0 \in \alpha'$, and that the vertex v_0 was added to δ. Then, $\mathrm{lk}_2(\alpha', \beta')$ is equal to the number of times α' crosses over β'. This can be broken down into the sum over all the edges, i.e.

$$\mathrm{lk}_2(\alpha', \beta') = \sum_{e \in \alpha'} l_e \pmod 2$$
$$\mathrm{lk}_2(\delta, \gamma) = \sum_{e \in \delta} l_e \pmod 2$$

Where l_e stands for the number of crossings between e and β'. (Note that $\beta' = \gamma$). Now, the only difference in in the summation are what edges are being summed over. δ and α' differ only by a triangle with v_0, v_i, and v_j as the vertices.

Proof of Lemma 1, Cont.

From the remarks on the previous slide,

$$\begin{aligned}
\mathrm{lk}_2(\alpha', \beta') &= \mathrm{lk}_2(\delta, \beta') + l_{(v_i, v_0)} + l_{(v_0, v_j)} - l_{(v_i, v_j)} \pmod 2 \\
&= \mathrm{lk}_2(\delta, \beta') + l_{(v_i, v_0)} + l_{(v_0, v_j)} + l_{(v_i, v_j)} \pmod 2 \\
&= \mathrm{lk}_2(\delta, \beta') + \mathrm{lk}_2(\Delta_{0ij}; \beta') \pmod 2
\end{aligned}$$

Now, if we sum over all possible choices of where v_0 can go, i.e. summing over a particular equivalence class of $[(\alpha, \beta)]$, then we see that each $l_{(v_0, v_i)}$ get duplicated, and cancel.

Proof of Lemma 1, Cont.

Next, the $l_{(v_i,v_j)}$, once summed over every edge in δ, is simply $\mathrm{lk}_2(\delta,\gamma)$, and once we sum over every in γ, we get another copy of $\mathrm{lk}_2(\delta,\gamma)$. Thus,

$$\sum_e l_e = 2\,\mathrm{lk}_2(\delta,\gamma) \pmod 2$$

And

$$\begin{aligned}\sum_{(\alpha,\beta)\in\Gamma_n\setminus\Delta(v_0)} \mathrm{lk}_2(\alpha,\beta) &= (n-1)\sum_{[(\alpha,\beta)]} \mathrm{lk}_2(\gamma,\delta) + 2\sum_{[(\alpha,\beta)]} \mathrm{lk}_2(\gamma,\delta) \\ &= (n+1)\sum_{[(\alpha,\beta)]} \mathrm{lk}_2(\gamma,\delta) \pmod 2\end{aligned}$$

as desired.

Symmetry yrtemmyS

We've divided Γ_n into $\Gamma_n \setminus \Delta(v_0)$ and $\Delta(v_0)$. We're going to further divide $\Delta(v_0)$ into more subsets, using symmetry.

To this end, we introduce a family of functions that maps the vertices of the spatial embedding to the other vertices

$$\tau_k(v_i) = \begin{cases} v_i, & i \leq k \\ v_{n-i} & k < i < n-k \\ v_i & i \geq n-k \end{cases}$$

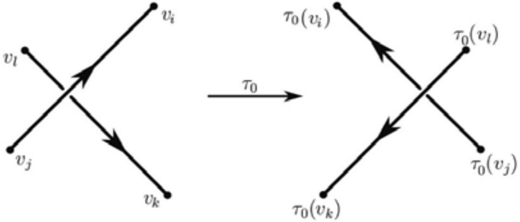
This induces a corresponding map on the edges.

Wait, How was that Symmetry?

Our choice of embedding also plays into this. Each map can be described as keeping the first k and last k vertices in the same places, and reflecting the other vertices over the line through v_0 and the center of the n-gon.

In particular, τ_0 is literally the reflection about this line, and it preserves crossings (That are not incident v_0): let e, f be edges, with $e = (v_i, v_j)$, $i < j$ and $f = (v_k, v_l)$, $k < l$.

If they intersect, then either $i < k < j < l$ or $k < i < l < j$. So, $n - l < n - j < n - k < n - i$ or $n - k < n - l < n - i < n - k$, which implies $\tau(e)$ and $\tau(f)$ intersect.

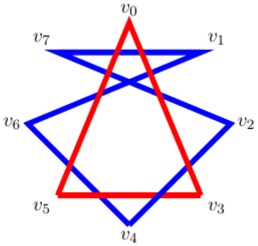


St_{τ_0} and St_τ

Now, we can define the two following notions: $St_\tau \subset St_{\tau_0} \subset \Delta(v_0)$

Definition

St_{τ_0} consists of elements $(\alpha, \beta) \in \Delta(v_0)$ so that $\tau_0(\alpha) = \alpha$ and $\tau_0(\beta) = \beta$



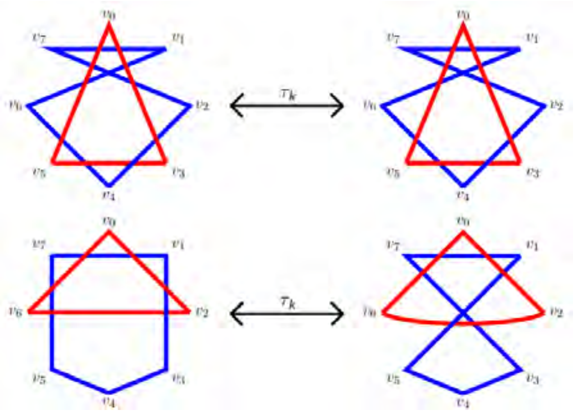
St_τ

Let $(\alpha, \beta) \in St_{\tau_0}$, and assume WLOG that α is the triangle containing v_0. Since it is invariant over τ, the other two vertices of the triangle must be v_k and v_{n-k}. This means that α is automatically invariant over τ_k.

Definition

St_τ consists of elements $(\alpha, \beta) \in St_{\tau_0}$ so that $\tau_k(\beta) = \beta$

St_τ



Dealing with $\Delta(v_0)$

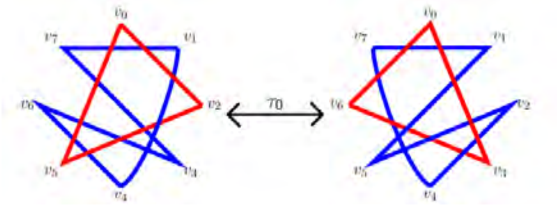
Just as we divided Γ into $\Gamma \setminus \Delta(v_0)$ and $\Delta(v_0)$, our plan for dealing with $\Delta(v_0)$ will be similar: We're going to divide $\Delta(v_0)$ into

$$\Delta(v_0) \setminus St_{\tau_0}, \quad St_{\tau_0} \setminus St_{\tau}, \quad \text{and } St_{\tau}$$

Since we're calculating things $\pmod 2$, the lack of symmetry in $\Delta(v_0) \setminus St_{\tau_0}$ and $St_{\tau_0} \setminus St_{\tau}$ will help us pair linking numbers, while the restrictiveness of St_{τ} will help us in its case.

Dealing with $\Delta(v_0) \setminus St_{\tau_0}$

Since we are excluding St_{τ_0}, $(\alpha, \beta) \neq (\tau_0(\alpha), \tau_0(\beta))$.
Since $\mathrm{lk}_2(\alpha, \beta) = \mathrm{lk}_2(\tau_0(\alpha), \tau_0(\beta))$, their sum is $0 \pmod 2$.

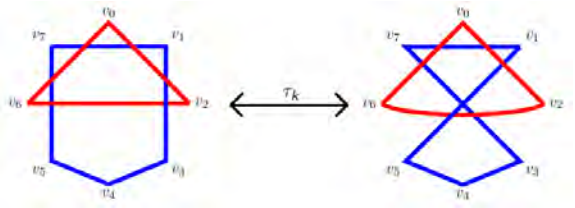


Thus

$$\sum_{(\alpha,\beta)\in\Delta(v_0)\setminus St_{\tau_0}} \mathrm{lk}_2(\alpha, \beta) = 0 \pmod 2.$$

Dealing with $St_{\tau_0} \setminus St_{\tau}$

Since we are excluding St_{τ}, $\beta \neq \tau_k(\beta)$
Since $\text{lk}_2(\alpha, \beta) = \text{lk}_2(\tau_0(\alpha), \tau_0(\beta))$, their sum is $0 \pmod 2$.

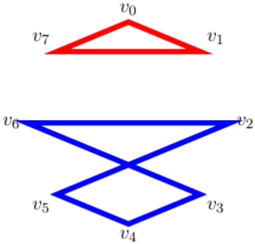


Thus

$$\sum_{(\alpha,\beta)\in St_{\tau_0}\setminus St_{\tau}} \text{lk}_2(\alpha, \beta) = 0 \pmod 2.$$

Dealing with St_τ

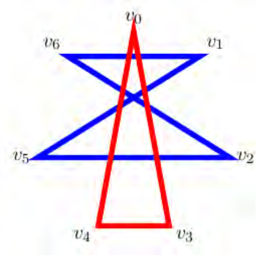
Finally, we must sum up $\mathrm{lk}_2(\alpha, \beta)$ for all (α, β) in St_τ.
First, consider the case where α uses vertices v_0, v_1, and v_{n-1}.



Clearly $\mathrm{lk}_2(\alpha, \beta) = 0$ for any (α, β) in this case.

Dealing with St_τ

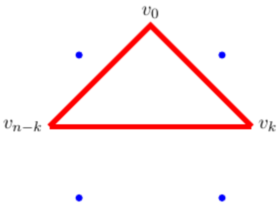
Now, consider the other case. Then α uses vertices v_0, v_k, and v_{n-k} with $k > 1$.
Suppose there are no vertices between v_k and v_{n-k}.



Clearly $\text{lk}_2(\alpha, \beta) = 0$ for any (α, β) in this case as well.

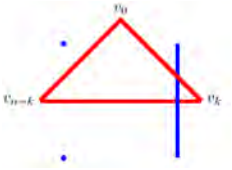
Dealing with St_τ

Now, suppose there are at least 2 vertices between v_k and v_{n-k}.

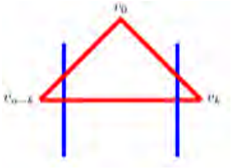


Dealing with St_τ

There must be at least one edge connecting the lower vertices to the upper ones:

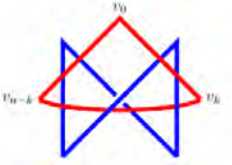


Due to τ_0 symmetry, we can add a second edge:



Dealing with St_τ

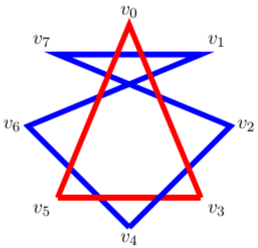
Due to τ_k symmetry, we can add two more edges:



After this, we have already closed off β. Therefore this is the only (α, β) in St_τ that has 2 or more vertices between v_k and v_{n-k}. And since this (α, β) has $\mathrm{lk}_2(\alpha, \beta) = 0$, we can also ignore this case.

Dealing with St_τ

Finally, we have the case where there is exactly 1 vertex between v_k and v_{n-k}. For every (α, β) in this case, $\text{lk}_2(\alpha, \beta) = 1$.

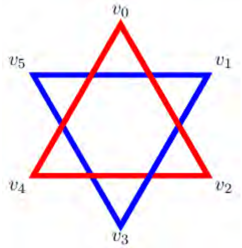


Now the question is: For each n, how many (α, β) are there of this type?

Dealing with St_τ

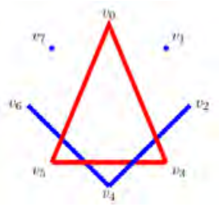
Clearly there are zero such (α, β) for any odd n, since in order to have a single point at the bottom of the n-gon, n must be even.

If $n = 6$, then there is exactly one such (α, β), shown below:



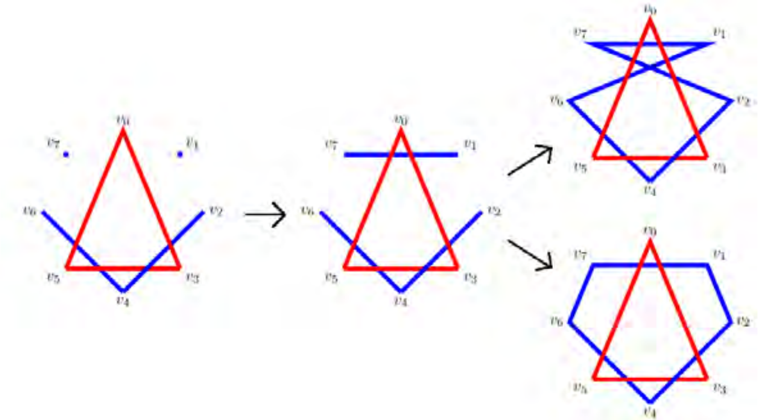
Dealing with St_τ

Finally, if $n > 6$ is even, then after adding the two symmetric edges of β to the bottom vertex there will be at least 2 more vertices that still need to be added to β.



After these remaining vertices are all connected with a symmetric path, there are two ways to connect the two parts of β.
This means that there are an even number of (α, β) in this case.

Dealing with St_T



Dealing with St_τ

To summarize, there are 4 cases for (α, β) in St_τ:

- α uses v_0, v_1, and v_{n-1}
- 0 vertices between v_k and v_{n-k}
- 2+ vertices between v_k and v_{n-k}
- Exactly 1 vertex between v_k and v_{n-k}

The sum of linking numbers mod 2 is always zero in the first 3 cases, and for the fourth case, the sum of linking numbers is 1 for $n = 6$ and 0 for all $n > 6$.

Land Ho! Eternity! Ashore At Last

Combining everything from above, for $n \geq 7$,

$$\begin{aligned}
\Omega(K_n) &= \sum_{(\alpha,\beta)\in\Gamma_n} \mathrm{lk}_2(\alpha,\beta) \pmod 2 \\
&= \sum_{(\alpha,\beta)\in\Gamma_n\setminus\Delta(v_0)} \mathrm{lk}_2(\alpha,\beta) + \sum_{(\alpha,\beta)\in\Delta(v_0)\setminus St_{\tau_0}} \mathrm{lk}_2(\alpha,\beta) \\
&\quad + \sum_{(\alpha,\beta)\in St_{\tau_0}\setminus St_{\tau}} \mathrm{lk}_2(\alpha,\beta) + \sum_{(\alpha,\beta)\in St_{\tau}} \mathrm{lk}_2(\alpha,\beta) \pmod 2 \\
&= (n+1)\sum_{[(\alpha,\beta)]} \mathrm{lk}_2(\gamma,\delta) + 0 + 0 + 0 \pmod 2 \\
&= (n+1)\Omega(K_{n-1}) \pmod 2
\end{aligned}$$

The End

$$\Omega(K_n) = (n+1)\Omega(K_{n-1}) \pmod 2$$

For odd n, $n + 1 = 0 \pmod 2$, so clearly $\Omega(K_n) = 0$.

For all even $n > 6$, $\Omega(K_{n-1}) = 0$ since $n - 1$ is odd.

Thus $\Omega(K_n) = 0$ for all $n > 6$, completing the proof.